Computing Virtual Classes of Representation Varieties using Topological Quantum Field Theories

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Representation varieties

X = connected closed manifold

$\pi_1(X)$ = fundamental group

G = algebraic group over k

<u>G-representation variety of X</u> $\quad \mathfrak{X}_G(X) = \text{Hom}(\pi_1(X), G)$

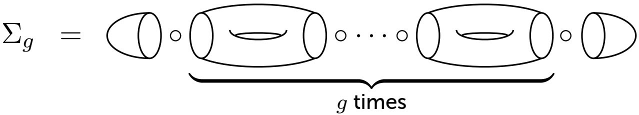
Non-abelian Hodge theory

When C complex projective curve

$$\begin{array}{ccc} & \mathcal{M}_{\mathsf{B}}(C,G) & \\ \text{Riemann–Hilbert} \swarrow\!\!\!\nearrow & & \nwarrow\!\!\!\searrow \text{Hitchin–Kobayashi} \\ \mathcal{M}_{\mathsf{DR}}(C,G) & \longleftrightarrow & \mathcal{M}_{\mathsf{Dol}}(C,G) \end{array}$$

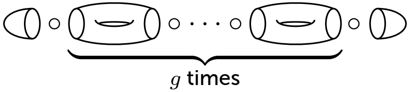
E-polynomial $\quad e(X) = \sum_{k,p,q} (-1)^k \; h_c^{k;p,q}(X) \; u^p v^q \in \mathbb{Z}[u, v]$

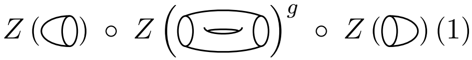
Idea: cut manifold in pieces and 'compute invariant piecewise'



Compute E-polynomial using Topological Quantum Field Theory

that is, a monoidal functor Z : **Bord** $\to$ R-**Mod**

From $\Sigma_g =$  we obtain

$$e(\mathfrak{X}_G(\Sigma_g)) = \frac{1}{e(G)^g}$$ 

Computed $Z\Big($  $\Big) : Z\Big($  $\Big) \to Z\Big($  $\Big)$ for $G = \mathbb{U}_2, \mathbb{U}_3, \mathbb{U}_4$

$\mathbb{U}_n = \{$upper triangular $n \times n$ matrices over $\mathbb{C}\}$

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For $G = \mathbb{U}_2$, we have

$Z\Big($  $\Big) = \begin{bmatrix} q^3 (q-1)^5 & q^3 (q-2)(q-1)^4 \\ q^3 (q-2)(q-1)^5 & q^3 (q-1)^4 \left(q^2 - 3q + 3\right) \end{bmatrix}$

For $G = \mathbb{U}_3$, we have

$$\begin{bmatrix}
(q-1)^2(q^2+q-1) & q^2(q-2)^2 & q^2(q-2)(q-1) & q^2(q-2)(q-1) & (q-1)^3(q+1) \\
q^3(q-2)^2(q-1)^2 & q^3(q^2-3q+3)^2 & q^3(q-2)(q-1)(q^2-3q+3) & q^3(q-2)(q-1)(q^2-3q+3) & q^3(q-2)^2(q-1)^2 \\
q^3(q-2)(q-1)^2 & q^3(q-2)(q^2-3q+3) & q^3(q-1)(q^2-3q+3) & q^3(q-2)^2(q-1) & q^3(q-2)(q-1)^2 \\
q^3(q-2)(q-1)^2 & q^3(q-2)(q^2-3q+3) & q^3(q-2)^2(q-1) & q^3(q-1)(q^2-3q+3) & q^3(q-2)(q-1)^2 \\
(q-1)^4(q+1) & q^2(q-2)^2(q-1) & q^2(q-2)(q-1)^2 & q^2(q-2)(q-1)^2 & (q-1)^2(q^3-q^2+1)
\end{bmatrix}$$

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For $G = \mathbb{U}_4$, we have

[illegible]

Theorem [Vogel, Hablicsek, arXiv:2008.06679]

Let $q = [\mathbb{A}^1_{\mathbb{C}}]$ be the class of the affine line in the Grothendieck ring of varieties. Then for all $g \geq 0$

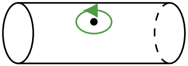
- $[\mathfrak{X}_{\mathbb{U}_2}(\Sigma_g)] = q^{2g-1}(q-1)^{2g+1}((q-1)^{2g-1}+1)$

- $$[\mathfrak{X}_{\mathbb{U}_3}(\Sigma_g)] = q^{3g-3}(q-1)^{2g}\left(q^2(q-1)^{2g+1} + q^{3g}(q-1)^2 + q^{3g}(q-1)^{4g} + 2q^{3g}(q-1)^{2g+1}\right)$$

- $$\begin{aligned}[\mathfrak{X}_{\mathbb{U}_4}(\Sigma_g)] = {} & q^{8g-2}\left(q-1\right)^{4g+2} + q^{8g-2}\left(q-1\right)^{6g+1} \\ & + q^{10g-4}\left(q-1\right)^{2g+3} + q^{10g-4}\left(q-1\right)^{4g+1}\left(2q^2-6q+5\right)^g \\ & + 3q^{10g-4}\left(q-1\right)^{4g+2} + q^{10g-4}\left(q-1\right)^{6g+1} + q^{12g-6}\left(q-1\right)^{8g} \\ & + q^{12g-6}\left(q-1\right)^{2g+3} + 3q^{12g-6}\left(q-1\right)^{4g+2} + 3q^{12g-6}\left(q-1\right)^{6g+1}\end{aligned}$$

Extensions

- Add parabolic data



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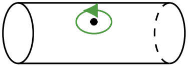
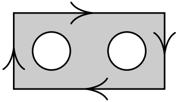


- Non-orientable surfaces



Theorem [Vogel, arXiv:2009.12310]

Let $N_r = \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ (r times). Then for all $r > 0$

- $[\mathfrak{X}_{\mathbb{U}_2}(N_r)] = 4q^{r-1}(q-1)^{2r-2} + 2q^{r-1}(q-1)^r$
- $$[\mathfrak{X}_{\mathbb{U}_3}(N_r)] = 4q^{2r-1}(q-1)^{2r-1} + 2q^{3r-3}(q-1)^{r+1} + 8q^{3r-3}(q-1)^{2r-1} + 8q^{3r-3}(q-1)^{3r-3}$$

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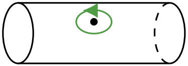

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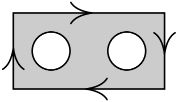

- Stacky TQFT [work in progress]

$$[\mathfrak{X}_G(\Sigma_g)/G]$$